Invariance Principles for Tempered Fractionally Integrated Processes

Farzad Sabzikar[1] and Donatas Surgailis[2]

March 8, 2017

[1]Iowa State University and [2]Vilnius University

Abstract

We discuss invariance principles for autoregressive tempered fractionally integrated moving averages in α-stable ($1 < \alpha \le 2$) i.i.d. innovations and related tempered linear processes with vanishing tempering parameter $\lambda \sim \lambda_*/N$. We show that the limit of the partial sums process takes a different form in the weakly tempered ($\lambda_* = 0$), strongly tempered ($\lambda_* = \infty$), and moderately tempered ($0 < \lambda_* < \infty$) cases. These results are used to derive the limit distribution of the OLS estimate of AR(1) unit root with weakly, strongly, and moderately tempered moving average errors.

Keywords: invariance principle; tempered linear process; autoregressive fractionally integrated moving average; tempered fractional stable/Brownian motion; tempered fractional unit root distribution;

1 Introduction

The present paper discusses partial sums limits and invariance principles for tempered moving averages

$$X_{d,\lambda}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} b_d(k)\zeta(t-k), \qquad t \in \mathbb{Z} \tag{1.1}$$

in i.i.d. innovation process $\{\zeta(t)\}$ with coefficients $b_d(k)$ regularly varying at infinity as k^{d-1}, viz.

$$b_d(k) \ \sim \ \frac{c_d}{\Gamma(d)}\, k^{d-1}, \qquad k \to \infty, \qquad c_d \neq 0, \quad d \neq 0 \tag{1.2}$$

where $d \in \mathbb{R}$ is a real number, $d \neq -1, -2, \ldots$ and $\lambda > 0$ is tempering parameter. In addition to (1.2) we assume that

$$\sum_{k=0}^{\infty} k^j b_d(k) = 0, \quad 0 \le j \le [-d], \quad -\infty < d < 0, \tag{1.3}$$

$$\sum_{k=0}^{\infty} |b_d(k)| < \infty, \quad \sum_{k=0}^{\infty} b_d(k) \neq 0, \qquad d = 0 \tag{1.4}$$

An important example of such processes is the two-parametric class ARTFIMA$(0, d, \lambda, 0)$ of tempered fractionally integrated processes, generalizing the well-known ARFIMA$(0, d, 0)$ class, written as

$$X_{d,\lambda}(t) = (1 - \mathrm{e}^{-\lambda} B)^{-d} \zeta(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \omega_{-d}(k) \zeta(t-k), \qquad t \in \mathbb{Z} \tag{1.5}$$

with coefficients given by power expansion $(1 - \mathrm{e}^{-\lambda} z)^{-d} = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \omega_{-d}(k) z^k, |z| < 1$, where $Bx(t) = x(t-1)$ is the backward shift. Due to the presence of the exponential tempering factor $\mathrm{e}^{-\lambda k}$ the series in (1.1) and (1.5) absolutely converges a.s. under general assumptions on the innovations, and defines a strictly stationary process. On the other hand, for $\lambda = 0$ the corresponding stationary processes in (1.1) and (1.5) exist under additional conditions on the parameter d. See Granger and Joyeux [12], Hosking [13], Brockwell and Davis [5], Kokoszka and Taqqu [15]. We also note (see e.g. [10], Ch. 3.2) that the (untempered) linear process $X_{d,0}$ of (1.1) with coefficients satisfying (1.2) for $0 < d < 1/2$ is said long memory, while (1.2) and (1.3) for $-1/2 < d < 0$ is termed negative memory and (1.4) short memory, respectively, parameter d usually referred to as memory parameter.

The model in (1.5) appeared in Giraitis et al. [8], which noted that for small $\lambda > 0$, $X_{d,\lambda}$ has a covariance function which resembles the covariance function of a long memory model for arbitrary large number of lags but eventually decays exponentially fast. [8] termed such behavior 'semi long-memory' and noted that it may have empirical relevance for modelling of financial returns. Giraitis et al. [9] propose the semi-long memory ARCH(∞) model as a contiguous alternative to (pure) hyperbolic and exponential decay which are often very hard to distinguish between in a finite sample. On the other side, Meerschaert et al. [20] effectively apply ARTFIMA$(0, d, \lambda, 0)$ in (1.5) for modeling of turbulence in the Great Lakes region.

The present paper obtains limiting behavior of tempered linear processes in (1.1) with small tempering parameter $\lambda = \lambda_N \to 0$ tending to zero together with the sample size. The important statistic is the partial sums process

$$S_N^{d,\lambda}(t) := \sum_{k=1}^{[Nt]} X_{d,\lambda}(k), \qquad t \in [0, 1] \tag{1.6}$$

of $X_{d,\lambda}$ in (1.1) with i.i.d. innovations $\{\zeta(t)\}$ in the domain of attraction of α-stable law, $1 < \alpha \le 2$. Functional limit theorems for the partial sums process play a crucial role in the R/S analysis, unit root testing, change-point analysis and many other time series inferences. See Lo [17], Phillips [21], Giraitis et al. [9], Lavancier et al. [16] and the references therein.

We prove that the limit behavior of (1.6) essentially depends on how fast $\lambda = \lambda_N$ tends to 0. Assume that there exists the limit

$$\lim_{N \to \infty} N \lambda_N = \lambda_* \in [0, \infty]. \tag{1.7}$$

Depending on the value of λ_*, the process X_{d,λ_N} will be called *strongly tempered* if $\lambda_* = \infty$, *weakly tempered* if $\lambda_* = 0$, and *moderately tempered* if $0 < \lambda_* < \infty$. While the behavior of S_N^{d,λ_N} in the strongly and weakly tempered cases is typical for short memory and long memory processes, respectively, the moderately tempered decay $\lambda_N \sim \lambda_*/N, \lambda_* \in (0,\infty)$ leads to *tempered fractional stable motion of second kind* (TFSM II) $Z^{II}_{H,\alpha,\lambda_*}, H = d + 1/\alpha > 0$ defined as a stochastic integral

$$Z^{II}_{H,\alpha,\lambda}(t) \quad := \quad \int_{\mathbb{R}} h_{H,\alpha,\lambda}(t;y) M_\alpha(\mathrm{d}y), \qquad t \in \mathbb{R} \tag{1.8}$$

with respect to α-stable Lévy process M_α with integrand

$$\begin{aligned} h_{H,\alpha,\lambda}(t;y) \quad &:= \quad (t-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(t-y)_+} - (-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(-y)_+} \\ &+ \quad \lambda \int_0^t (s-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(s-y)_+} \mathrm{d}s, \qquad y \in \mathbb{R}. \end{aligned} \tag{1.9}$$

TFSM II and its Gaussian counterpart *tempered fractional Brownian motion of second kind* (TFBM II) were recently introduced in Sabzikar and Surgailis [22], the above processes being closely related to the *tempered fractional stable motion* (TFSM) and the *tempered fractional Brownian motion* (TFBM) defined in Meerschaert and Sabzikar [19] and Meerschaert and Sabzikar [18], respectively. As shown in [22], TFSM and TFSM II are different processes, especially striking are their differences as $t \to \infty$.

As an application of our invariance principles we obtain the limit distribution of the OLS estimator $\widehat{\beta}_N$ of the slope parameter in AR(1) model with tempered ARTFIMA$(0,d,\lambda_N,0)$ errors and small tempering parameter $\lambda_N \to 0$ satisfying (1.7), under the null (unit root) hypothesis $\beta = 1$. In the case of (untempered) ARFIMA$(0,d,0)$ error process with finite variance and standardized i.i.d. innovations, Sowell [24] proved that the distribution of the normalized statistic $N^{1\wedge(1+2d)}(\widehat{\beta}_N - 1)$ tends to the so-called *fractional unit root distribution* written in terms of fractional Brownian motion with parameter $H = d + \frac{1}{2}$. Sowell's [24] result extends the classical unit root distribution for weakly dependent errors in Phillips [21] to fractionally integrated error process, yielding drastically different limits for $0 < d < 1/2, d = 0$ and $-1/2 < d < 0$.

It turns out that in the case of ARTFIMA$(0,d,\lambda_N,0)$ error process with $\lambda_N \sim \lambda_*/N$, the limit distribution of $\widehat{\beta}_N$ depends on $\lambda_* \in [0,\infty]$ and d. Roughly speaking (see Theorem 5.2 for precise formulation), in the moderately tempered case $0 < \lambda_* < \infty$ the limit distribution of $\widehat{\beta}_N$ writes similarly to Sowell [24] with FBM B_H replaced by TFBM II B^{II}_{H,λ_*} and the convergence holds for all $-1/2 < d < \infty$ in contrast to [24] which is limited to $|d| < 1/2$. Under strong tempering $\lambda_N/N \to \lambda_* = \infty$, the limit distribution of $\widehat{\beta}_N$ is written in terms of standard Brownian motion but takes a different form in the cases $d > 0, d = 0$ and $d < 0, d \neq \mathbb{N}_-$; moreover, except for the i.i.d. case $d = 0$, this limit is different from Sowell's limit in [24] and also from the unit root distribution in Dickey and Fuller [6] and Phillips [21].

The paper is organized as follows. Section 2 introduces ARTFIMA(p, d, λ, q) class and provides basic properties of these processes. In Section 3 we define TFSM II/TFBM II. Section 4 contains the main results of the paper (invariance principles). Section 5 discusses the application to unit root testing. The proofs of the main results are relegated to Section 6.

In what follows, C denotes generic constants which may be different at different locations. We write $\xrightarrow{\mathrm{d}}$, $\stackrel{\mathrm{d}}{=}$, $\xrightarrow{\mathrm{fdd}}$, $\stackrel{\mathrm{fdd}}{=}$ for the weak convergence and equality of distributions and finite-dimensional distributions. $\mathbb{N}_\pm := \{\pm 1, \pm 2, \dots\}$, $\mathbb{R}_+ := (0,\infty)$, $(x)_\pm := \max(\pm x, 0), x \in \mathbb{R}$, $\int := \int_{\mathbb{R}}$. $L^p(\mathbb{R})\,(p \ge 1)$ denotes the Banach space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ with finite norm $\|f\|_p = \big(\int |f(x)|^p \mathrm{x}\big)^{1/p}$.

2 Tempered fractionally integrated process

In this section, we define ARTFIMA(p, d, λ, q) process and discuss its basic properties. Let $\Phi(z) = 1 - \sum_{i=1}^p \phi_i z^i$ and $\Theta(z) = 1 + \sum_{i=1}^q \theta_i z^i$ be polynomials with real coefficients of degree $p, q \ge 0$, such that $\Phi(z)$ does not vanish on $\{z \in \mathbb{C}, |z| \le 1\}$ and $\Phi(z)$ and $\Theta(z)$ have no common zeros. Let $d \in \mathbb{R} \setminus \mathbb{N}_+$. Consider Taylor's expansion

$$\frac{\Theta(z)}{\Phi(z)}(1-z)^d = \sum_{k=0}^\infty a_d(k) z^k, \qquad |z| < 1. \tag{2.1}$$

Note that

$$a_d(k) = \sum_{s=0}^k \omega_d(k)\psi(k-s), \qquad k \ge 0, \tag{2.2}$$

where $\omega_d(k) = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(d)}$, and $\psi(j)$ are the coefficients of the power series $\sum_{j=0}^\infty \psi(j) z^j = \Theta(z)/\Phi(z), |z| \le 1$. We use the fact (see Kokoszka and Taqqu ([15], Lemma 3.1) that for any $d \in \mathbb{R} \setminus \mathbb{N}_-$

$$\omega_{-d}(k) = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)} \ = \ \Gamma(d)^{-1} k^{d-1}\big(1 + O(1/k)\big), \quad k \to \infty. \tag{2.3}$$

Proposition 2.1 *Let $\Theta(1) \ne 0$. For any $d \in \mathbb{R} \setminus \mathbb{N}_-$, the coefficients $a_{-d}(k), k \ge 0$ satisfy conditions* (1.2)-(1.4). *In particular, for any $d \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$*

$$a_{-d}(k) \sim \frac{\Theta(1)}{\Phi(1)\Gamma(d)} k^{d-1}\Big(1 + O\big(\frac{1}{k}\big)\Big), \qquad k \to \infty. \tag{2.4}$$

Proof. Let us prove (2.4). By definition, $a_{-d}(k) = \sum_{j=0}^k \psi(j)\omega_{-d}(k-j), k \ge 0$, see (2.2). It is well-known that $|\psi(j)| \le C\mathrm{e}^{-cj}$ for some constants $C, c > 0$, see ([15], proof of Lemma 3.2). Note $\Theta(1)/\Phi(1) = \sum_{j=0}^\infty \psi(j) \ne 0$. We have

$$\begin{aligned}
&\big|a_{-d}(k) - (\Theta(1)/\Phi(1))\omega_{-d}(k)\big| \\
&= \ \Big|\sum_{j=0}^k \psi(j)\big(\omega_{-d}(k-j) - \omega_{-d}(k)\big) - \omega_{-d}(k)\sum_{j>k}\psi(j)\Big| \ \le \ \sum_{i=1}^3 \ell_{k,i},
\end{aligned}$$

where

$$\begin{aligned}\ell_{k,1} &:= \sum_{0\le j\le k^{1/4}} |\psi(j)|\big|\omega_{-d}(k-j)-\omega_{-d}(k)\big|,\\ \ell_{k,2} &:= \sum_{k^{1/4}<j\le k} |\psi(j)|\big(|\omega_{-d}(k-j)|+|\omega_{-d}(k)|\big),\end{aligned}$$

and $\ell_{k,3} := |\omega_{-d}(k)|\sum_{j>k}|\psi(j)| \le Ck^{d-1}\sum_{j>k}\mathrm{e}^{-cj} = o(k^{d-2})$ since $\psi(j)$ decay exponentially. Similarly, $\ell_{k,2} \le C\mathrm{e}^{-ck^{1/4}}k\max_{1\le j\le k}|\omega_{-d}(j)| = o(k^{d-2})$. Using (2.3) we obtain $|\ell_{k,1}| \le C(\ell'_{k,1} + \ell''_{k,2})$, where $\ell''_{k,2} := k^{d-2}\sum_{j\ge 0}|\psi(j)| = O(k^{d-2})$ and

$$\begin{aligned}\ell'_{k,1} &:= \sum_{0\le j\le k^{1/4}} |\psi(j)|\big(k^{d-1}-(k-j)^{d-1}\big) = k^{d-1}\sum_{0\le j\le k^{1/2}}|\psi(j)|\Big(1-(1-\frac{j}{k})^{d-1}\Big)\\ &\le Ck^{d-2}\sum_{j\ge 0} j|\psi(j)| = O(k^{d-2})\end{aligned}$$

proving $\big|a_{-d}(k)-(\Theta(1)/\Phi(1))\omega_{-d}(k)\big| = O(k^{d-2})$ and hence (2.4) in view of (2.3). Thus, $a_{-d}(k)$, $d\ne 0$ satisfy (1.2). Condition (1.4) is obvious from $a_0(k)=\psi(k)$ and properties of $\psi(k)$ stated above.

It remains to prove (1.3). Let $j < -d < j+1$ for some $j = 0,1,\dots$. Then since $\Psi(z) := \Theta(z)/\Phi(z)$ is analytic on $\{z\in\mathbb{C}, |z|\le 1+\delta\}$ $(\exists\,\delta>0)$ and $|a_{-d}(k)|\le Ck^{d-1}$, see (2.4), the function $\sum_{k=0}^\infty a_{-d}(k)z^k = \Psi(z)(1-z)^{-d}$ is j times differentiable on $\{|z|\le 1\}$ and $\frac{\partial^i}{\partial z^i}\Psi(z)(1-z)^{-d}\big|_{z=1} = \sum_{r=0}^i\binom{i}{r}\frac{\partial^r\Psi(z)}{\partial z^r}\big|_{z=1}\frac{\partial^{i-r}(1-z)^{-d}}{\partial z^{i-r}}\big|_{z=1} = 0, 0\le i\le j$. Hence, $0=\sum_{k=i}^\infty k(k-1)\cdots(k-i+1)a_{-d}(k) = \sum_{k=0}^\infty k(k-1)\cdots(k-i+1)a_{-d}(k)$ for any $0\le i\le j$, proving (1.3) and the proposition, too. $\Box$

Definition 2.2 *Let the autoregressive polynomials $\Theta(z),\Phi(z)$ of degree p,q satisfy the above conditions, and $d\in\mathbb{R}\setminus\mathbb{N}_-, \lambda\ge 0$. Moreover, let $\zeta=\{\zeta(t), t\in\mathbb{Z}\}$ be a stationary process with $\mathbb{E}|\zeta(0)|<\infty$. By ARTFIMA$(p,d,\lambda,q)$ process with innovation process ζ we mean a stationary moving-average process $X_{p,d,\lambda,q}=\{X_{p,d,\lambda,q}(t), t\in\mathbb{Z}\}$ defined by*

$$X_{p,d,\lambda,q}(t) = \sum_{k=0}^\infty \mathrm{e}^{-\lambda k}a_{-d}(k)\zeta(t-k), \qquad t\in\mathbb{Z} \tag{2.5}$$

where the series converges in L_1.

Remark 2.3 (i) For $\lambda=0$ and $|d|<1/2$ and i.i.d. innovations with zero mean and unit variance, ARTFIMA$(p,d,0,q)$ process $X_{p,d,0,q}$ coincides with ARFIMA(p,d,q) process, see e.g. Brockwell and Davis [5]. Particularly, $X_{d,0}=X_{0,d,0,0}$ in (2.5) is a stationary solution of the AR(∞) equation

$$(1-B)^dX_{d,0}(t) = \sum_{j=0}^\infty \omega_d(j)X_{d,0}(t-j) = \zeta(t) \tag{2.6}$$

and the series in (2.5) and (2.6) converge in L_2, meaning that $X_{d,0}$ is invertible.

(ii) For $\lambda = 0$ and zero mean i.i.d. α-stable innovations, $1 < \alpha < 2$, the definition of $X_{p,d,0,q}$ in Definition 2.2 agrees with the definition of ARFIMA(p, d, q) process in [15], who showed that the series in (2.5) converges a.s. and in L_1 for $d < 1 - \frac{1}{\alpha}$.

Proposition 2.4 *Let $\Theta(z), \Phi(z)$ satisfy the above conditions and $\lambda > 0$. Then the series in* (2.5) *converges in L_1 for any $d \in \mathbb{R} \setminus \mathbb{N}_-$ hence ARTFIMA(p, d, λ, q) process $X_{p,d,\lambda,q}$ in Definition 2.2 is well-defined for arbitrary (stationary) innovation process ζ with finite mean. Moreover, if $|\Theta(z)| > 0$, $|z| \le 1$ then $X_{p,d,\lambda,q}$ is invertible:*

$$\zeta(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \widetilde{a_d}(k) X_{p,d,\lambda,q}(t-k), \tag{2.7}$$

where

$$\sum_{k=0}^{\infty} \widetilde{a_d}(k)\ z^k = \frac{\Phi(z)}{\Theta(z)}(1-z)^d, \qquad |z| < 1 \tag{2.8}$$

and the series in (2.7) *converges in L_1.*

Proof. The convergence in L_1 of the series in (2.5) follows from (2.4). To show invertibility of these series, note that by Proposition 2.1 the coefficients in (2.8) satisfy the bound $|\widetilde{a_d}(k)| \le Ck^{-d-1}, k \ge 1$ for $d \in \mathbb{R} \setminus \mathbb{N}_+$, and an exponential bound $|\widetilde{a_d}(k)| \le C\mathrm{e}^{-ck}, k \ge 1, c > 0$ for $d \in \mathbb{N}_+$. Hence, $\mathrm{e}^{-\lambda k}|\widetilde{a_d}(k)| \le C\mathrm{e}^{-\lambda k}k^{-d-1}$, for any $d \in \mathbb{R}$ implying the convergence in L_1 of the series in (2.7). Finally, equality in (2.7) follows from identity $1 = \big(\frac{\Phi(z)}{\Theta(z)}(1-z)^d\big)\big(\frac{\Theta(z)}{\Phi(z)}(1-z)^{-d}\big), |z| < 1$. □

Proposition 2.5 describes some second order properties of ARTFIMA(p, d, λ, q) with standardized innovations.

Proposition 2.5 L*et $X_{p,d,\lambda,q}$ be ARTFIMA(p, d, λ, q) process in* (2.5)*, $d \in \mathbb{R} \setminus \mathbb{N}_-, \lambda > 0$, with standardized i.i.d. innovations $\{\zeta(t), t \in \mathbb{Z}\}, \mathbb{E}\zeta(0) = 0, \mathbb{E}\zeta^2(0) = 1$. Then $\mathbb{E}X_{p,d,\lambda,q}(t) = 0, \mathbb{E}X^2_{p,d,\lambda,q}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-2\lambda k} a^2_{-d}(k) < \infty$ and*

(i) The spectral density of $X_{p,d,\lambda,q}$ is given by

$$h(x) \ = \ \frac{1}{2\pi}\left|\frac{\Theta(\mathrm{e}^{-\imath x})}{\Phi(\mathrm{e}^{-\imath x})}\right|^2 (1 - 2\mathrm{e}^{-\lambda}\cos x + \mathrm{e}^{-2\lambda})^{-d}, \qquad -\pi \le x \le \pi.$$

(ii) The covariance function of $X_{0,d,\lambda,0}$ is given by

$$\gamma_{d,\lambda}(k) = \mathbb{E}X_{0,d,\lambda,0}(0)X_{0,d,\lambda,0}(k) = \frac{\mathrm{e}^{-\lambda k}\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)}\ {}_2F_1(d, k+d; k+1; \mathrm{e}^{-2\lambda}), \tag{2.9}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function (see e.g. [11]). Moreover,

$$\sum_{k\in\mathbb{Z}} \big|\gamma_{d,\lambda}(k)\big| < \infty, \qquad \sum_{k\in\mathbb{Z}} \gamma_{d,\lambda}(k) = (1-\mathrm{e}^{-\lambda})^{-2d} \tag{2.10}$$

and

$$\gamma_{d,\lambda}(k) \sim Ak^{d-1}\mathrm{e}^{-\lambda k}, \quad k \to \infty, \quad \textit{where} \quad A = (1-\mathrm{e}^{-2\lambda})^{-d}\Gamma(d)^{-1}. \tag{2.11}$$

Proof. (i) From the transfer function $(\Theta(\mathrm{e}^{-\imath x})/\Phi(\mathrm{e}^{-\imath x}))(1-\mathrm{e}^{-\imath x-\lambda})^{-d}$ of the filter in (2.5) we have that $h(x) = \frac{1}{2\pi}|\Theta(\mathrm{e}^{-\imath x})/\Phi(\mathrm{e}^{-\imath x})|^2|1-\mathrm{e}^{-\imath x-\lambda}|^{-2d}$, where $|1-\mathrm{e}^{-\imath x-\lambda}|^2 = 1-2\mathrm{e}^{-\lambda}\cos(x)+\mathrm{e}^{-2\lambda}$, proving part (i). (ii) (2.9) follows from $\gamma_{d,\lambda}(k) = \int_{-\pi}^{\pi}\cos(kx)h(x)\,\mathrm{d}x$ and ([11], Eq. 9.112). The first relation in (2.10) follows from $\sum_{j\in\mathbb{Z}}|e^{-\lambda j}\omega_d(j)| < \infty$, see (2.3), and the second one from $\sum_{k\in\mathbb{Z}}\gamma_{d,\lambda}(k) = 2\pi h(0) = (1-\mathrm{e}^{-\lambda})^{-2d}$. Finally, (2.11) is proved in ([8], (4.15)). □

3 Tempered fractional Brownian and stable motions of second kind

This section contains the definition of TFBM II/TFSM II and some of its properties from Sabzikar and Surgailis [22]. The reader is referred to the aforementioned paper for further properties of these processes including relation to tempered fractional calculus, relation between TFBM II/TFSM II and TFBM/TFSM, dependence properties of the increment process (tempered fractional Brownian/stable noise), local and global asymptotic self-similarity.

For $1 < \alpha \le 2$, let $M_\alpha = \{M_\alpha(t), t\in\mathbb{R}\}$ be an α-stable Lévy process with stationary independent increments and characteristic function

$$\mathbb{E}\mathrm{e}^{\imath\theta M_\alpha(t)} = \mathrm{e}^{-\sigma^\alpha|\theta|^\alpha|t|\left(1-\imath\beta\tan(\pi\alpha/2)\mathrm{sign}(\theta)\right)}, \qquad \theta\in\mathbb{R}, \tag{3.1}$$

where $\sigma > 0$ and $\beta\in[-1,1]$ are the scale and skewness parameters, respectively. For $\alpha = 2$, $M_2(t) = \sqrt{2}\sigma B(t)$, where B is a standard Brownian motion with variance $\mathbb{E}B^2(t) = t$. Stochastic integral $I_\alpha(f) \equiv \int f(x)M_\alpha(\mathrm{d}x)$ is defined for any $f\in L^\alpha(\mathbb{R})$ as α-stable random variable with characteristic function

$$\mathbb{E}\mathrm{e}^{\imath\theta I_\alpha(f)} = \exp\{-\sigma^\alpha|\theta|^\alpha\int|f(x)|^\alpha\big(1-\imath\beta\tan(\pi\alpha/2)\mathrm{sign}(\theta f(x))\big)\mathrm{d}x\}, \quad \theta\in\mathbb{R}. \tag{3.2}$$

see e.g. [23, Chapter 3].

For $t\in\mathbb{R}, H>0, 1<\alpha\le 2, \lambda\ge 0$ consider the function $y\mapsto h_{H,\alpha,\lambda}(t;y):\mathbb{R}\to\mathbb{R}$ given in (1.9). Note $h_{H,\alpha,\lambda}(t;\cdot)\in L^\alpha(\mathbb{R})$ for any $t\in\mathbb{R}, \lambda>0, 1<\alpha\le 2, H>0$ and also for $\lambda=0, 1<\alpha\le 2, H\in(0,1)$. We will use the following integral representation of (1.9) (see [22]).

For $H > \frac{1}{\alpha}$:

$$h_{H,\alpha,\lambda}(t;y) = (H-\tfrac{1}{\alpha})\int_0^t(s-y)_+^{H-\frac{1}{\alpha}-1}\mathrm{e}^{-\lambda(s-y)_+}\mathrm{d}s. \tag{3.3}$$

For $0 < H < \frac{1}{\alpha}$:

$$h_{H,\alpha,\lambda}(t;y) = (H-\tfrac{1}{\alpha})\begin{cases}\int_0^t(s-y)_+^{H-\frac{1}{\alpha}-1}\mathrm{e}^{-\lambda(s-y)_+}\mathrm{d}s, & y<0,\\ -\int_t^\infty(s-y)_+^{H-\frac{1}{\alpha}-1}\mathrm{e}^{-\lambda(s-y)_+}\mathrm{d}s + \lambda^{\frac{1}{\alpha}-H}\Gamma(H-\frac{1}{\alpha}), & y\ge 0.\end{cases} \tag{3.4}$$

Definition 3.1 Let M_α be α-stable Lévy process in (3.1), $1 < \alpha \le 2$ and $H > 0$, $\lambda > 0$. Define, for $t \in \mathbb{R}$, the stochastic integral

$$Z^{II}_{H,\alpha,\lambda}(t) \quad := \quad \int h_{H,\alpha,\lambda}(t; y) M_\alpha(\mathrm{d}y). \tag{3.5}$$

The process $Z^{II}_{H,\alpha,\lambda} = \{Z^{II}_{H,\alpha,\lambda}(t), t \in \mathbb{R}\}$ will be called *tempered fractional stable motion of second kind* (TFSM II). A particular case of (3.5) corresponding to $\alpha = 2$

$$B^{II}_{H,\lambda}(t) \quad := \quad \frac{1}{\Gamma(H + \frac{1}{2})} \int h_{H,2,\lambda}(t; y) B(\mathrm{d}y) \tag{3.6}$$

will be called *tempered fractional Brownian motion of second kind* (TFBM II).

The next proposition states some basic properties of TFSM II $Z^{II}_{H,\alpha,\lambda}$.

Proposition 3.2 ([22])

(i) $Z^{II}_{H,\alpha,\lambda}$ *in* (3.5) *is well-defined for any* $t \in \mathbb{R}$ *and* $1 < \alpha \le 2, H > 0, \lambda > 0$, *as a stochastic integral in* (3.2).

(ii) $Z^{II}_{H,\alpha,\lambda}$ *in* (3.5) *has stationary increments and α-stable finite-dimensional distributions. Moreover, it satisfies the following scaling property:* $\big\{Z^{II}_{H,\alpha,\lambda}(bt), t \in \mathbb{R}\big\} \overset{\mathrm{fdd}}{=} \big\{b^H Z^{II}_{H,\alpha,b\lambda}(t), t \in \mathbb{R}\big\}$, $\forall\ b > 0$.

(iii) $Z^{II}_{H,\alpha,\lambda}$ *in* (3.5) *has a.s. continuous paths if either* $\alpha = 2, H > 0$, *or* $1 < \alpha < 2, H > 1/\alpha$ *hold.*

4 Invariance principles

In this section, we discuss invariance principles and the convergence of (normalized) partial sums process

$$S^{d,\lambda}_N(t) := \sum_{k=1}^{[Nt]} X_{d,\lambda}(k), \qquad t \in [0,1] \tag{4.1}$$

of tempered linear process $X_{d,\lambda}$ in (1.1) with i.i.d. innovations belonging to the domain of attraction of α-stable law, $1 < \alpha \le 2$ (see below). We shall assume that the tempering parameter $\lambda \equiv \lambda_N \to 0$ as $N \to \infty$ and following limit exists:

$$\lim_{N \to \infty} N \lambda_N = \lambda_* \in [0, \infty]. \tag{4.2}$$

Recall that X_{d,λ_N} is said *strongly tempered* if $\lambda_* = \infty$, *weakly tempered* if $\lambda_* = 0$, and *moderately tempered* if $0 < \lambda_* < \infty$. We show that the limits of the partial sums process in (4.1) exist under condition (4.2) and depend on λ_*, d, α; moreover, in all cases these limits belong to the class of TFSM II processes defined in Section 3.

Definition 4.1 *Write $\zeta \in D(\alpha)$, $1 < \alpha \le 2$ if*

(i) $\alpha = 2$ *and* $\mathbb{E}\zeta = 0$, $\sigma^2 := \mathbb{E}\zeta^2 < \infty$, *or*

(ii) $1 < \alpha < 2$ *and there exist some constants* $c_1, c_2 \ge 0, c_1 + c_2 > 0$ *such that* $\lim_{x\to\infty} x^\alpha \mathrm{P}(\zeta > x) = c_1$ *and* $\lim_{x\to-\infty} |x|^\alpha \mathrm{P}(\zeta \le x) = c_2$; *moreover,* $\mathbb{E}\zeta = 0$.

Condition $\zeta \in D(\alpha)$ implies that r.v. ζ belongs to the domain of normal attraction of an α-stable law. In other words, if $\zeta(i), i \in \mathbb{Z}$ are i.i.d. copies of ζ then

$$N^{-1/\alpha} \sum_{i=1}^{[Nt]} \zeta(i) \xrightarrow{\mathrm{fdd}} M_\alpha(t), \qquad N \to \infty, \tag{4.3}$$

where M_α is an α-stable Lévy process in (3.1) with σ, β determined by c_1, c_2, see ([7], pp. 574-581). We shall use the following criterion for convergence of weighted sums in i.i.d. r.v.s. See ([10], Prop. 14.3.2), [2], [14].

Proposition 4.2 *Let* $1 < \alpha \le 2$ *and* $Q(g_N) = \sum_{t\in\mathbb{Z}} g_N(t)\zeta(t)$ *be a linear form in i.i.d. r.v.s* $\zeta(t) \in D(\alpha)$ *with real coefficients* $g_N(t), t \in \mathbb{Z}$. *Assume that there exists* $p \in [1, \alpha)$ *if* $\alpha < 2$, $p = 2$ *if* $\alpha = 2$ *and a function* $g \in L^p(\mathbb{R})$ *such that the functions*

$$\widetilde{g}_N(x) := N^{1/\alpha} g_N([xN]), \qquad x \in \mathbb{R} \tag{4.4}$$

satisfy

$$\|\widetilde{g}_N - g\|_p \to 0 \qquad (N \to \infty). \tag{4.5}$$

Then $Q(g_N) \xrightarrow{\mathrm{d}} \int g(x) M_\alpha(\mathrm{d}x)$, *where* M_α *is as in* (4.3) *and* (3.1).

Write $\xrightarrow{D[0,1]}$ for weak convergence of random processes in the Skorohod space $D[0,1]$ equipped with J_1-topology, see [4].

In Theorem 4.3 below, X_{d,λ_N} is a tempered linear process of (1.1) with i.i.d. innovations $\zeta(t) \in D(\alpha), 1 < \alpha \le 2$, coefficients $b_d(k), k \ge 0$, $d \in \mathbb{R} \setminus \mathbb{N}_-$ satisfying (1.2)-(1.4), and tempering parameter $0 < \lambda_N \to 0\,(N \to \infty)$ satisfying (4.2). W.l.g., we shall assume that the asymptotic constant c_d in (1.2)-(1.4) equals 1: $c_d = 1\ \forall d \in \mathbb{R} \setminus \mathbb{N}_-$.

Theorem 4.3 *(i)* (Strongly tempered process.) *Let* $\lambda_* = \infty$ *and* $d \in \mathbb{R} \setminus \mathbb{N}_-$. *Then*

$$N^{-\frac{1}{\alpha}} \lambda_N^d S_N^{d,\lambda_N}(t) \xrightarrow{\mathrm{fdd}} M_\alpha(t), \tag{4.6}$$

where M_α *is* α*-stable Lévy process in* (4.3). *Moreover, if* $\alpha = 2$ *and* $\mathbb{E}|\zeta(0)|^p < \infty$ *for some* $p > 2$ *then*

$$N^{-\frac{1}{2}} \lambda_N^d S_N^{d,\lambda_N}(t) \xrightarrow{D[0,1]} \sigma B(t), \tag{4.7}$$

where B is a standard Brownian motion and $\sigma > 0$ some constant.

(ii) (Weakly tempered process.) *Let $\lambda_* = 0$ and $H = d + \frac{1}{\alpha} \in (0,1)$. Then*

$$N^{-H} S_N^{d,\lambda_N}(t) \quad \xrightarrow{\text{fdd}} \quad \Gamma(d+1)^{-1} Z_{H,\alpha,0}(t), \tag{4.8}$$

where $Z_{H,\alpha,0}$ is a linear fractional α-stable motion, see Definition 3.1. Particularly, for $\alpha = 2$, $Z_{H,2,0}$ is a multiple of FBM.
Moreover, if either $1 < \alpha \le 2, 1/\alpha < H < 1$, or $\alpha = 2, 0 < H < 1/2$ and $\mathbb{E}|\zeta(0)|^p < \infty\,(\exists p > 1/H)$ hold, then $\xrightarrow{\text{fdd}}$ in (4.8) can be replaced by $\xrightarrow{D[0,1]}$.

(iii) (Moderately tempered process.) *Let $\lambda_* \in (0,\infty)$ and $H = d + \frac{1}{\alpha} > 0$. Then*

$$N^{-H} S_N^{d,\lambda_N}(t) \quad \xrightarrow{\text{fdd}} \quad \Gamma(d+1)^{-1} Z^{II}_{H,\alpha,\lambda_*}(t), \tag{4.9}$$

where $Z^{II}_{H,\alpha,\lambda_*}$ is a TFSM II as defined in Definition 3.1.
Moreover, if either $1 < \alpha \le 2, 1/\alpha < H$, or $\alpha = 2, 0 < H < 1/2$ and $\mathbb{E}|\zeta(0)|^p < \infty\,(\exists p > 1/H)$ hold, then $\xrightarrow{\text{fdd}}$ in (4.9) can be replaced by $\xrightarrow{D[0,1]}$.

Remark 4.4 Note that for $\lambda_N = \lambda_*/N$ the normalization in (4.6) becomes $N^{-(\frac{1}{\alpha}+d)}\lambda_*^d$ where the exponent $\frac{1}{\alpha} + d = H$ is the same as in (4.8) and (4.9).

Remark 4.5 The functional convergence in (4.6), case $1 < \alpha < 2$ (the case of discontinuous limit process) is open and apparently does not hold in the usual J_1-topology, see [3]. In the case of (4.8) and (4.9) and $1 < \alpha < 2, 0 < H < \frac{1}{\alpha}$, functional convergence cannot hold in principle since the limit processes do not belong to $D[0,1]$.

5 Tempered fractional unit root distribution

A fundamental problem of time series is testing for the unit root $\beta = 1$ in the AR(1) model

$$Y(t) = \beta Y(t-1) + X(t), \qquad t = 1,2,\dots,N, \quad Y(0) = 0 \tag{5.1}$$

with stationary error process $X = \{X(t), t \in \mathbb{Z}\}$. The classical approach to the unit root testing is based on the limit distribution of the OLS estimator $\widehat{\beta}_N$

$$\widehat{\beta}_N = \frac{\sum_{t=1}^N Y(t)Y(t-1)}{\sum_{t=1}^N Y^2(t-1)}. \tag{5.2}$$

The limit theory for $\widehat{\beta}_N$ in the case of weakly dependent errors X was developed in Phillips [21]. We note that [21] makes an extensive use of invariance principle for the error process. Sowell [24] obtained the limit distribution of $\widehat{\beta}_N$ in the case of strongly dependent ARFIMA$(0,d,0)$ error

process with finite variance and standardized i.i.d. innovations. [24] proved that the distribution of the normalized statistic $N^{1\wedge(1+2d)}(\widehat{\beta}_N - 1)$ tends to that of the ratio

$$\frac{1}{2\int_0^1 B_H^2(s)\,ds} \begin{cases} B_H^2(1), & 0 < d < 1/2, \\ B^2(1) - 1, & d = 0, \\ -H\Gamma(H+\frac{1}{2})/\Gamma(\frac{3}{2}-H), & -1/2 < d < 0, \end{cases} \tag{5.3}$$

where $H = d + \frac{1}{2}$ and B_H is a FBM with parameter $H \in (0,1)$, $B = B_{1/2}$ being a standard Brownian motion.

In this section we extend Sowell's [24] result to ARTFIMA$(0, d, \lambda_N, 0)$ error process with small tempering parameter $\lambda_N \sim \lambda_*/N \to 0$ as in (4.2). Although Theorem 5.2 can be generalized to more general tempered processes with finite variance as in Theorem 4.3, our choice of ARTFIMA$(0, d, \lambda_N, 0)$ as the error process is motivated by better comparison to [24]. As noted in Section 1, the degree of tempering has a strong effect on the limit distribution of $\widehat{\beta}_N$ and leads to a new two-parameter family of tempered fractional unit root distributions. Following [24], we decompose

$$\widehat{\beta}_N - 1 = \widehat{A}_N - \widehat{B}_N, \tag{5.4}$$

where

$$\widehat{A}_N := \frac{Y^2(N)}{2\sum_{t=1}^N Y^2(t-1)}, \qquad \widehat{B}_N := \frac{\sum_{t=1}^N X^2(t)}{2\sum_{t=1}^N Y^2(t-1)}. \tag{5.5}$$

Under the unit root hypothesis $\beta = 1$ we have $Y(t) = \sum_{i=1}^t X(i) = S_N(t/N)$, where $S_N(x) := \sum_{t=1}^{[Nx]} X(t), x \in [0,1]$ is the partial sums process. Particularly, the statistics in (5.5) can be rewritten as

$$\widehat{A}_N = \frac{S_N^2(1)}{2N\int_0^1 S_N^2(s)\,ds}, \qquad \widehat{B}_N = \frac{\sum_{t=1}^N X^2(t)}{2N\int_0^1 S_N^2(s)\,ds}. \tag{5.6}$$

For ARTFIMA error process X_{d,λ_N}, the behavior of $S_N^2(1)$ and $\int_0^1 S_N^2(s)\,ds$ can be derived from Theorem 4.3. The behavior of $\sum_{t=1}^N X^2(t)$ is established in the following proposition.

Proposition 5.1 *Let X_{d,λ_N} be an ARTFIMA$(0, d, \lambda_N, 0)$ process in* (2.5) *with i.i.d. innovations* $\{\zeta(t)\}, \mathbb{E}\zeta(0) = 0, \mathbb{E}\zeta^2(0) = 1$, *fractional parameter $d \in \mathbb{R} \setminus \mathbb{N}_-$ and tempering parameter $\lambda_N \to 0$. Moreover, let* $\mathbb{E}|\zeta(0)|^p < \infty\, (\exists\, p > 2)$. *Then*

$$\frac{1}{N}\sum_{t=1}^N X_{d,\lambda_N}^2(t) \xrightarrow{p} \frac{\Gamma(1-2d)}{\Gamma^2(1-d)}, \qquad d < 1/2, \tag{5.7}$$

$$\frac{\lambda_N^{2d-1}}{N}\sum_{t=1}^N X_{d,\lambda_N}^2(t) \xrightarrow{p} \frac{\Gamma(d-1/2)}{2\sqrt{\pi}\,\Gamma(d)}, \qquad d > 1/2, \tag{5.8}$$

$$\frac{1}{N|\log\lambda_N|}\sum_{t=1}^N X_{d,\lambda_N}^2(t) \xrightarrow{p} \frac{1}{\pi}, \qquad d = 1/2. \tag{5.9}$$

The main result of this section is the following theorem.

Theorem 5.2 *Consider the AR(1) model in* (5.1) *with* $\beta = 1$ *and ARTFIMA*$(0, d, \lambda_N, 0)$ *error process* $X = \{X_{d,\lambda_N}(t)\}$ *in* (2.5) *with i.i.d. innovations* $\{\zeta(t), t \in \mathbb{Z}\}, \mathbb{E}\zeta(0) = 0, \mathbb{E}\zeta^2(0) = 1, \mathbb{E}|\zeta(0)|^p < \infty$ $(\exists\, p > 2 \vee 1/(d+1/2))$*, fractional parameter* $d \in \mathbb{R} \setminus \mathbb{N}_-$ *and tempering parameter* $\lambda_N > 0$ *satisfying* (4.2).

(i) (Strongly tempered errors.) *Let* $\lambda_* = \infty$, $d \in \mathbb{R} \setminus \mathbb{N}_-$. *Then*

$$\min(1, \lambda_N^{-2d}) N(\widehat{\beta}_N - 1) \quad \xrightarrow{\mathrm{d}} \quad \frac{1}{2\int_0^1 B^2(s)\, \underset{\cdot}{s}} \begin{cases} B^2(1), & d > 0, \\ B^2(1) - 1, & d = 0, \\ -\Gamma(1-2d)/\Gamma(1-d)^2, & d < 0, \end{cases}$$

where B *is a standard Brownian motion.*

(ii) (Weakly tempered errors.) *Let* $\lambda_* = 0$ *and* $H = d + \frac{1}{2} \in (0, 1)$. *Then*

$$N^{1\wedge(1+2d)}(\hat{\beta}_N - 1) \quad \xrightarrow{\mathrm{d}} \quad \frac{1}{2\int_0^1 B_H^2(s) \underset{\cdot}{s}} \begin{cases} B_H^2(1), & \frac{1}{2} < H < 1, \\ B^2(1) - 1, & H = \frac{1}{2} \\ -H\Gamma(H + \frac{1}{2})/\Gamma(\frac{3}{2} - H), & 0 < H < \frac{1}{2}, \end{cases} \tag{5.10}$$

where B_H *is a FBM with variance* $\mathbb{E}B_H^2(t) = t^{2H}, B = B_{1/2}$.

(iii) (Moderately tempered errors.) *Let* $0 < \lambda_* < \infty$ *and* $H = d + \frac{1}{2} > 0$. *Then*

$$N^{1\wedge(1+2d)}(\hat{\beta}_N - 1) \quad \xrightarrow{\mathrm{d}} \quad \frac{1}{2\int_0^1 \big(B_{H,\lambda_*}^{II}(s)\big)^2 \underset{\cdot}{s}} \begin{cases} (B_{H,\lambda_*}^{II}(1))^2, & H > \frac{1}{2}, \\ (B_{\frac{1}{2},\lambda_*}^{II}(1))^2 - 1, & H = \frac{1}{2} \\ -\Gamma(2(1-H))/\Gamma(\frac{3}{2} - H)^2, & 0 < H < \frac{1}{2}, \end{cases} \tag{5.11}$$

where $B_{H,\lambda}^{II}$ *is a TFBM II given by* (3.6).

Remark 5.3 The limit (5.10) in the weakly tempered case coincides with Sowell's limit (5.3). Since $B_{H,0}^{II} \overset{\mathrm{fdd}}{=} C_1 B_H$, where $C_1^2 = \Gamma(1-H)/2^{2H} H\Gamma(H+1/2)\sqrt{\pi}$, see [22], the r.v. on the r.h.s. of (5.11) for $\lambda_* = 0, 0 < H < 1$ also coincides with (5.3), however the convergence (5.11) holds for any $H > 0$ in contrast to $H \in (0, 1)$ in (5.10).

Proof of Theorem 5.2. From Theorem 4.3 and Proposition 5.1 we obtain the joint convergence of

$$\Big(a_N (S_N^{d,\lambda_N}(1))^2,\ a_N \int_0^1 (S_N^{d,\lambda_N}(s))^2 \underset{\cdot}{s},\ b_N \sum_{t=1}^N X_{d,\lambda_N}^2(t)\Big) \tag{5.12}$$

where $a_N \to 0, b_N \to 0$ are normalizations defined in these theorems and depending on d and λ_*, in each case (i)-(iii) of Theorem 5.2. Then the statement of Theorem 5.2 follows from (5.12), the continuous mapping theorem and the representation of $\widehat{\beta}_N - 1$ in (5.4)-(5.6) through the corresponding quantities in (5.12). $\square$

6 Proofs of Theorem 4.3 and Proposition 5.1

Proof of Theorem 4.3. (i) We restrict the proof of finite-dimensional convergence in (4.6) to one-dimensional convergence at $t>0$ since the general case follows similarly. We use Proposition 4.2. Accordingly, write $N^{-\frac{1}{\alpha}}\lambda_N^d S_N^{d,\lambda_N}(t) = Q(g_N(t,\cdot)) = \sum_{i\in\mathbb{Z}} g_N(t;i)\zeta(i)$, where $g_N(t;i) := N^{-1/\alpha}\lambda_N^d \sum_{k=1\vee i}^{[Nt]} \mathrm{e}^{-\lambda_N(k-i)}\, b_d(k-i)$. It suffices to prove (4.5) for suitable p and $g(t;x) := \mathbf{1}_{[0,t]}(x)$. We have

$$\widetilde{g}_N(t;x) = \lambda_N^d \sum_{k=1\vee[Nx]}^{[Nt]} \mathrm{e}^{-\lambda_N(k-[Nx])}\, b_d(k-[Nx]). \tag{6.1}$$

Let us prove the point-wise convergence:

$$\widetilde{g}_N(t;x) \to g(t;x) = \mathbf{1}_{[0,t]}(x), \quad \forall\, x \neq 0, t. \tag{6.2}$$

Let us prove that conditions (1.2)-(1.4) imply that

$$G_N := \lambda_N^d \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_N k}\, b_d(k) \to 1 \qquad (N\to\infty). \tag{6.3}$$

First, let $d>0$. Then since $\sum_{k=0}^n b_d(k) \sim (1/d\Gamma(d))n^d, n\to\infty$ according to (1.2), then by applying the Tauberian theorem for power series (Feller [7], Ch. 13, § 5, Thm. 5) we have $\sum_{k=0}^\infty b_d(k)\,\mathrm{e}^{-\lambda_N k} \sim (1-\mathrm{e}^{-\lambda_N})^{-d}$, $N\to\infty$, proving (6.3) for $d>0$. Next, let $d=0$. Then in view of (1.4) the dominated convergence theorem applies yielding $\sum_{k=0}^\infty \mathrm{e}^{-\lambda_N k}\, b_0(k) \to \sum_{k=0}^\infty b_0(k) = 1$ and (6.3) follows again.

Next, let $-1<d<0$. Then $\tilde{b}_d(k) := \sum_{i=k}^\infty b_d(i) \sim (-1/d\Gamma(d))k^{\tilde{d}-1}$, $k\to\infty, \tilde{d} := d+1 \in (0,1)$, $\tilde{b}_d(0)=0$ and $\sum_{k=0}^\infty \mathrm{e}^{-\lambda k} b_d(k) = -\mathrm{e}^{\lambda}(1-\mathrm{e}^{-\lambda})\sum_{k=1}^\infty \mathrm{e}^{-\lambda k}\tilde{b}_d(k)$ using summation by parts. Then the aforementioned Tauberian theorem implies $\sum_{k=0}^\infty \mathrm{e}^{-\lambda_N k} b_d(k) \sim (1-\mathrm{e}^{-\lambda_N})^{1-\tilde{d}} = (1-\mathrm{e}^{-\lambda_N})^{-d}$ proving (6.3) for $-1<d<0$. In the general case $-j<d<-j+1, j=1,2,\ldots$ relation (6.3) follows similarly using summation by parts j times.

Let $0<x<t$ first. Then $\widetilde{g}_N(t;x) = G_N - \widetilde{g}_N^*(t;x)$, where

$$\widetilde{g}_N^*(t;x) := \lambda_N^d \sum_{k>[Nt]-[Nx]} b_d(k)\mathrm{e}^{-\lambda_N k}.$$

Using (1.2) for $d\neq 0$ we obtain

$$\begin{aligned}
|\widetilde{g}_N^*(t;x)| &\le C(N\lambda_N)^d N^{-1} \sum_{k>[Nt]-[Nx]} \mathrm{e}^{-(N\lambda_N)(k/N)}(k/N)^{d-1} \\
&\le C(N\lambda_N)^d \int_{t-x}^\infty \mathrm{e}^{-N\lambda_N y} y^{d-1} \mathrm{d}y \\
&= C\int_{(t-x)(N\lambda_N)}^\infty \mathrm{e}^{-z} z^{d-1} \mathrm{d}z \to 0
\end{aligned} \tag{6.4}$$

since $N\lambda_N \to \infty$. A similar result for $d = 0$ follows directly from (1.4). In view of (6.3), this proves (6.2) for $0 < x < t$. Next, let $x < 0$. Then similarly as above

$$
\begin{aligned}
\widetilde{g}_N(t;x) &\le C(N\lambda_N)^d N^{-1} \sum_{k>|[Nx]|} \mathrm{e}^{-(N\lambda_N)(k/N)} (k/N)^{d-1} \\
&\le C(N\lambda_N)^d \int_{|x|}^{\infty} \mathrm{e}^{-N\lambda_N y} y^{d-1} \underset{\cdot}{\mathrm{y}} \\
&= C \int_{|x|(N\lambda_N)}^{\infty} \mathrm{e}^{-z} z^{d-1} \underset{\cdot}{\mathrm{z}} \to 0,
\end{aligned}
\tag{6.5}
$$

proving (6.2). Note also that $|\widetilde{g}_N(t;x)| \le C\lambda_N^d \sum_{k=0}^{\infty} |b_d(k)| \le C\lambda_N^d \le C$ for $d < 0$ and $|\widetilde{g}_N(t;x)| \le C(N\lambda_N)^d \int_0^\infty \mathrm{e}^{-N\lambda_N y}\ y^{d-1} \underset{\cdot}{\mathrm{y}} \le C \int_0^\infty \mathrm{e}^{-z} z^{d-1} \underset{\cdot}{\mathrm{z}} \le C$ for $d > 0$, implying that $|\widetilde{g}_N(t;x)|$ is bounded uniformly in $x \in \mathbb{R}, N \ge 1$; moreover, according to (6.5) $|\widetilde{g}_N(t;x)| \le C\mathrm{e}^{-c'|x|}, x < -2$ decays exponentially with $x \to -\infty$ with some $c' > 0$ uniformly in $N \ge 1$. This proves (4.5) and hence (4.6).

Consider the functional convergence in (4.7). This follows from the tightness criterion

$$
N^{-p/2} \lambda_N^{pd} \mathbb{E}|S_N^{d,\lambda_N}(t) - S_N^{d,\lambda_N}(s)|^p \le C|L_N(t) - L_N(s)|^{p/2}, \qquad \forall\ 0 \le s < t \le 1, \tag{6.6}
$$

where $L_N(t) := [Nt]/N$, see [4], also ([10], Lemma 4.4.1). By Rosenthal's inequality (see e.g. [10], Proposition 4.4.3), $\mathbb{E}|S_N^{d,\lambda_N}(t) - S_N^{d,\lambda_N}(s)|^p \le C\mathbb{E}^{p/2}|S_N^{d,\lambda_N}(t) - S_N^{d,\lambda_N}(s)|^2$ where $\mathbb{E}|S_N^{d,\lambda_N}(t) - S_N^{d,\lambda_N}(s)|^2 = N\lambda_N^{-2d} \int |\widetilde{g}_N(t;x) - \widetilde{g}_N(s;x)|^2 \underset{\cdot}{\mathrm{x}} \le CN\lambda_N^{-2d}|L_N(t) - L_N(s)|$ follows similarly as above, proving (6.6) and part (i), too.

(ii) Relation (4.8) is well-known with $S_N^{d,\lambda_N}(t)$ replaced by $S_N^{d,0}(t)$, see e.g. [2], also ([10], Cor. 4.4.1), so that it suffices to prove

$$
R_N(t) := S_N^{d,\lambda_N}(t) - S_N^{d,0}(t) = o_p(N^H). \tag{6.7}
$$

With Proposition 4.2 in mind, (6.7) follows from $\|\widetilde{g}_N^0(t;\cdot)\|_p \to 0$, where

$$
\begin{aligned}
|\widetilde{g}_N^0(t;x)| &:= N^{-d} \Big| \sum_{k=1\vee[Nx]}^{[Nt]} b_d(k - [Nx]) \left(1 - \mathrm{e}^{-\lambda_N(k-[Nx])}\right) \Big| \\
&\le C(N\lambda_N) \frac{1}{N^{d+1}} \sum_{k=1\vee[Nx]}^{[Nt]} (k - [Nx])^d \le C(N\lambda_N) \to 0
\end{aligned}
$$

uniformly in $x \in \mathbb{R}$, where we used (1.2), inequality $1 - \mathrm{e}^{-x} \le x\,(x \ge 0)$ and the fact that $H \in (0,1), 1 < \alpha \le 2$ imply $-1 < d < 1 - \frac{1}{\alpha}$. For $x < -1$ a similar argument leads to

$$
\begin{aligned}
|\widetilde{g}_N^0(t;x)| &\le CN^{-d} \sum_{k=1}^{[Nt]} (k - [Nx])^{d-1} \ \le CN^{-d}\big(([Nt] - [Nx])^d - (-[Nx])^d\big) \\
&\le C((t-x)^d - (-x)^d) \le C(-x)^{d-1}
\end{aligned}
\tag{6.8}
$$

implying the dominating bound $|\widetilde{g}^0_N(t;x)| \le C/(1+|x|)^{1-d} =: \bar g(x)$ where $\|\bar g\|_p < \infty$ for $1 \le p < \alpha$ sufficiently close to α due to condition $d < 1 - \frac{1}{\alpha}$. This proves $\|\widetilde{g}^0_N(t;\cdot)\|_p \to 0$, hence (6.7) and (4.8), too.

To prove the tightness part of (ii), we use a similar criterion as in (6.6), viz.,

$$N^{-pH}\mathbb{E}|S^{d,\lambda_N}_N(t) - S^{d,\lambda_N}_N(s)|^p \le C|L_N(t) - L_N(s)|^q, \qquad \forall\ 0 \le s < t \le 1, \tag{6.9}$$

with $L_N(t) = [Nt]/N$ and suitable $p, q > 1$. Let first $\frac{1}{\alpha} < H < 1$, or $0 < d < 1 - \frac{1}{\alpha}$. Let $\widetilde{g}_N(t;x) = N^{-d}\sum_{k=1\vee[Nx]}^{[Nt]} b_d(k-[Nx])\,\mathrm{e}^{-\lambda_N(k-[Nx])}$. Then for $0 < s < t$

$$\begin{aligned}
|\widetilde{g}_N(t;x) - \widetilde{g}_N(s;x)| &\le N^{-d}\sum_{k=[Ns]+1}^{[Nt]} |b_d(k-[Nx])| \le CN^{-d}\sum_{k=[Ns]+1}^{[Nt]} (k-[Nx])_+^{d-1} \\
&\le CN^{-d}\int_{[Ns]}^{[Nt]} (y-[Nx])_+^{d-1}\underset{.}{\mathrm{y}} \\
&\le C((L_N(t)-L_N(x))_+^d - (L_N(s)-L_N(x))_+^d)
\end{aligned} \tag{6.10}$$

and therefore for $1 \le p < \alpha$ sufficiently close to α

$$\begin{aligned}
N^{-pH}\mathbb{E}|S^{d,\lambda_N}_N(t) - S^{d,\lambda_N}_N(s)|^p &\le C\|\widetilde{g}_N(t;\cdot) - \widetilde{g}_N(s;\cdot)\|_p^p \\
&\le C\int_0^\infty |(L_N(t)-L_N(s)+L_N(x))^d - L_N(x)^d|^p \underset{.}{\mathrm{x}} \\
&\le C\Big((L_N(t)-L_N(s))^{pd}N^{-1} + \int_0^\infty |(L_N(t)-L_N(s)+x)^d - x^d|^p \underset{.}{\mathrm{x}}\Big) \\
&\le C\big(L_N(t)-L_N(s)\big)^{1+pd}
\end{aligned} \tag{6.11}$$

since $L_N(t) - L_N(s) \ne 0$ implies $L_N(t) - L_N(s) \ge N^{-1}$. This proves (6.9) with $q = 1 + pd > 1$.

Next, let $\alpha = 2$ and $0 < H < \frac{1}{2}$, or $-1/2 < d < 0$. Then (6.9) holds for $S^{d,0}_N$ instead of S^{d,λ_N}_N with $q = pH > 1$, see ([10], proof of Prop. 4.4.4). Hence, it suffices to prove a similar bound for $R_N(t)$ in (6.7). By Rosenthal's inequality (see the proof (6.6)) $\mathbb{E}|R_N(t)-R_N(s)|^p \le C\mathbb{E}^{p/2}|R_N(t)-R_N(s)|^2$ and hence $N^{-pH}\mathbb{E}|R_N(t) - R_N(s)|^p \le C\|\widetilde{g}^0_N(t;\cdot) - \widetilde{g}^0_N(s;\cdot)\|_2^p$, where

$$\begin{aligned}
|\widetilde{g}^0_N(t;x) - \widetilde{g}^0_N(s;x)| &\le CN^{-d}\sum_{k=[Ns]+1}^{[Nt]} (k-[Nx])_+^{d-1}\big(1-\mathrm{e}^{-\lambda_N(k-[Nx])_+}\big) \\
&\le C\begin{cases} (-L_N(x))^{d-1}(L_N(t)-L_N(s)), & x < -1, \\ C\big((L_N(t)-L_N(x))_+^{d+1} - (L_N(s)-L_N(x))_+^{d+1}\big), & -1 < x < 1, \end{cases}
\end{aligned} \tag{6.12}$$

similarly as in (6.10). Hence $N^{-pH}\mathbb{E}|R_N(t) - R_N(s)|^p \le C\|\widetilde{g}^0_N(t;\cdot) - \widetilde{g}^0_N(s;\cdot)\|_2^p \le C(L_N(t) - L_N(s))^{1+p(1+d)}$ follows, proving (6.7) and part (ii), too.

(iii) Similarly as in the proof of (ii), let us prove $\|\widetilde{g}_N(t;\cdot) - g(t;\cdot)\|_p \to 0$, where

$$\widetilde{g}_N(t;x) := \tfrac{1}{N^d}\sum_{k=1\vee[Nx]}^{[Nt]} b_d(k-[Nx])\,\mathrm{e}^{-\lambda_N(k-[Nx])}, \quad g(t;x) := \tfrac{1}{\Gamma(1+d)}h_{H,\alpha,\lambda_*}(t;x), \tag{6.13}$$

see the definition of $h_{H,\alpha,\lambda}$ in (1.9). First, let $d > 0$ or $H > \frac{1}{\alpha}$. Then using (1.2) we obtain the point-wise convergence

$$
\begin{aligned}
\widetilde{g}_N(t;x) \quad &= \quad \frac{1}{N\Gamma(d)} \sum_{\frac{1}{N} \vee \frac{[Nx]}{N} \le \frac{k}{N} \le \frac{[Nt]}{N}} \big(\frac{k}{N} - \frac{[Nx]}{N}\big)^{d-1} \big(1 + \epsilon_{N1}(k,x)\big) \mathrm{e}^{-\lambda_*(\frac{k}{N} - \frac{[Nx]}{N})(1+\epsilon_{N2})} \\
&\to \quad \frac{1}{\Gamma(d)} \int_0^t (y-x)_+^{d-1} \mathrm{e}^{-\lambda_*(y-x)_+} dy \ = \ \frac{1}{\Gamma(1+d)} h_{H,\alpha,\lambda_*}(t;x), \quad \forall\ x \neq 0, t, \quad (6.14)
\end{aligned}
$$

see (3.3), where

$$
\epsilon_{N1}(k,x) := \Gamma(d)(k - [Nx])^{1-d} b_d(k - [Nx]) - 1 \to 0, \qquad \epsilon_{N2} := (N\lambda_N/\lambda_*) - 1 \to 0
$$

as $N \to \infty, k - [Nx] \to \infty$ and $|\epsilon_{N1}(k,x)| + |\epsilon_{N2}| < C$ is bounded uniformly in N, k, x. Therefore (6.14) holds by the dominated convergence theorem. We also have from (6.13) that $|\widetilde{g}_N(t;x)| \le CN^{-d} \sum_{k=1\vee[Nx]}^{[Nt]} (k - [Nx])^{d-1} \mathrm{e}^{-(\lambda_*/2)(k-[Nx])} \le C \int_0^t (s-x)_+^{d-1} \mathrm{e}^{-(\lambda_*/2)(s-x)_+} ds = Ch_{d+1/\alpha,\alpha,\lambda_*/2}(t;x) =: \bar{g}(x)$ is dominated by an integrable function, see (3.3), with $\|\bar{g}\|_p < \infty$. This proves (4.9) for $d > 0$.

Next, let $-\frac{1}{\alpha} < d < 0$. Decompose $\widetilde{g}_N(t;x)$ in (6.13) as $\widetilde{g}_N(t;x) = \widetilde{g}_{N1}(x) - \widetilde{g}_{N2}(t;x)$, where

$$
\begin{aligned}
\widetilde{g}_{N1}(x) \quad &:= \quad N^{-d} \sum_{k=1\vee[Nx]}^{\infty} b_d(k - [Nx])\, \mathrm{e}^{-\lambda_N(k-[Nx])}, \\
\widetilde{g}_{N2}(x) \quad &:= \quad N^{-d} \sum_{[Nt]+1}^{\infty} b_d(k - [Nx])\, \mathrm{e}^{-\lambda_N(k-[Nx])}.
\end{aligned}
$$

Let $0 < x < t$. First we have

$$
\widetilde{g}_{N1}(x) = \frac{\sum_{j=0}^{\infty} b_d(j)\, \mathrm{e}^{-\lambda_N j}}{\lambda_N^{-d}} \ (N\lambda_N)^{-d} \to \lambda_*^{-d}
$$

since the last ratio tends to 1 as $N \to \infty$, see (6.3). We also have

$$
\begin{aligned}
\widetilde{g}_{N2}(t;x) \quad &= \quad \frac{1}{N\Gamma(d)} \sum_{\frac{k}{N} > \frac{[Nt]}{N}} \big(\frac{k}{N} - \frac{[Nx]}{N}\big)^{d-1} \big(1 + \epsilon_{N1}(k,x)\big) \mathrm{e}^{-\lambda_*(\frac{k}{N} - \frac{[Nx]}{N})(1+\epsilon_{N2})} \\
&\to \quad \frac{1}{\Gamma(d)} \int_t^{\infty} (s-x)_+^{d-1} \mathrm{e}^{-\lambda_*(s-x)_+} ds \ = \ \lambda_*^{-d} - \frac{1}{\Gamma(1+d)} h_{H,\alpha,\lambda_*}(t;x),
\end{aligned}
$$

see (3.4), similarly to (6.14). This proves the point-wise convergence $\widetilde{g}_N(t;x) \to g(t;x) = \Gamma(1+d)^{-1} h_{H,\alpha,\lambda}(t;x)$ for $0 < x < t$ and the proof for $x < 0$ is similar. Then $\|\widetilde{g}_N(t;\cdot) - g(t;\cdot)\|_p \to 0$ or (4.9) for $-\frac{1}{\alpha} < d < 0$ follows similarly as in the case $d > 0$ above.

Consider the proof of tightness in (iii). We use the same criterion (6.9) as in part (ii). Let first $d > 0$. Then for $|x| \le 1$ the bound in (6.10) and hence $\int_{-1}^{1} |\widetilde{g}_N(t;x) - \widetilde{g}_N(s;x)|^p dx \le C(L_N(t) - L_N(s))^{1+pd}$ follows as in (6.11). On the other hand, for $x < -1$ we have

$$
|\widetilde{g}_N(t;x) - \widetilde{g}_N(s;x)| \quad \le \quad C\mathrm{e}^{-(\lambda_*/2)|x|} \int_{[Ns/N]}^{[Nt]/N} dy \ = \ C\mathrm{e}^{-(\lambda_*/2)|x|} (L_N(t) - L_N(s))
$$

implying $\int_{-\infty}^{-1} |\widetilde{g}_N(t;x) - \widetilde{g}_N(s;x)|^p \dot{x} \le C|L_N(t) - L_N(s)|^p$. Consequently, (6.9) for $d > 0$ holds with $q = (1+pd) \wedge p > 1$. Finally, (6.9) for $\alpha = 2, -1/2 < d < 0$ follows as in case (ii) since (6.12) holds in the case $\lambda_N = O(1/N)$ as well. This ends the proof of Theorem 4.3. $\square$

Proof of Proposition 5.1. (i) By stationarity, $N^{-1}\mathbb{E}\sum_{t=1}^N X_{d,\lambda_N}^2(t) = \mathbb{E}X_{d,\lambda_N}^2(0)$. Let us first prove the convergence of expectations:

$$
\begin{aligned}
\mathbb{E}X_{d,\lambda_N}^2(0) &\to \frac{\Gamma(1-2d)}{\Gamma^2(1-d)}, & d < 1/2, \qquad (6.15)\\
\lambda_N^{2d-1}\mathbb{E}X_{d,\lambda_N}^2(0) &\to \frac{\Gamma(d-1/2)}{2\sqrt{\pi}\,\Gamma(d)}, & d > 1/2, \qquad (6.16)\\
|\log \lambda_N|^{-1}\mathbb{E}X_{d,\lambda_N}^2(0) &\to \frac{1}{\pi}, & d = 1/2, \qquad (6.17)
\end{aligned}
$$

as $N \to \infty$. Since $\mathbb{E}X_{d,\lambda_N}^2(0) = \sum_{k=0}^\infty \mathrm{e}^{-2\lambda_N k}\omega_{-d}^2(k)$, with $\omega_{-d}(k) \equiv \omega_{-d}(k)$ defined in (2.3), and

$$
\sum_{k=0}^n \omega_{-d}^2(k) \sim \begin{cases} (1/(2d-1)\Gamma^2(d))n^{2d-1}, & d > 1/2,\\ (1/\Gamma^2(1/2))\log(n), & d = 1/2,\\ \sum_{k=0}^\infty \omega_{-d}^2(k) = \Gamma(1-2d)/\Gamma^2(1-d), & d < 1/2 \end{cases}
$$

as $n \to \infty$, the convergences in (6.15)-(6.17) follows from the Tauberian theorem in [7] used in the proof of Theorem 4.3 (i) above.

With (6.15)-(6.17) in mind, (5.7)-(5.9) follow from

$$
Q_N \equiv N^{-1}\sum_{t=1}^N \Big[X_{d,\lambda_N}^2(t) - \mathbb{E}X_{d,\lambda_N}^2(t)\Big] = \begin{cases} o_p(1), & d < 1/2,\\ o_p(\lambda_N^{1-2d}), & d > 1/2,\\ o_p(|\log \lambda_N|), & d = 1/2. \end{cases} \tag{6.18}
$$

Let $\omega_{-d,\lambda}(k) := \omega_{-d}(k)\mathrm{e}^{-\lambda k}$. We have $Q_N = Q_{N1} + Q_{N2}$, where

$$
\begin{aligned}
Q_{N1} &= N^{-1}\sum_{s \le N}(\zeta^2(s) - \mathbb{E}\zeta^2(s))\sum_{t=1\vee s}^N \omega_{-d,\lambda_N}^2(t-s),\\
Q_{N2} &= N^{-1}\sum_{s_2 < s_1 \le N}\zeta(s_1)\sum_{t=1\vee s_1}^N \omega_{-d,\lambda_N}(t-s_1)\omega_{-d,\lambda_N}(t-s_2)\zeta(s_2).
\end{aligned}
$$

Note $Q_{Ni}, i = 1,2$ are sums of martingale differences. We shall use the well-known moment inequality for sums of martingale differences:

$$
\mathbb{E}|\sum_{i\ge 1}\xi_i|^\alpha \le 2\sum_{i \ge 1}\mathbb{E}|\xi_i|^\alpha \tag{6.19}
$$

see e.g. ([10], Prop. 2.5.2), which is valid for any $1 \le \alpha \le 2$ and any sequence $\{\xi_i, i \ge 1\}$ with $\mathbb{E}|\xi_i|^\alpha < \infty, \mathbb{E}[\xi_i|\xi_j, 1 \le j < i] = 0, i \ge 1$.

First, let $d > 1/2$. Using $\mathbb{E}|\zeta(0)|^p < \infty, 2 < p < 4$ and (6.19) with $\alpha = p/2$ we obtain

$$
\begin{aligned}
\mathbb{E}|Q_{N1}|^{p/2} &\le CN^{-p/2} \sum_{s \le N} \Big| \sum_{t=1\vee s}^{N} \omega^2_{-d,\lambda_N}(t-s)\Big|^{p/2} \\
&\le CN^{-p/2} \sum_{s \le N} \Big(\sum_{t=1\vee s}^{N} (t-s)_+^{2(d-1)} \mathrm{e}^{-2\lambda_N(t-s)}\Big)^{p/2} \\
&\le CN^{-p/2} \Big\{ \int_N^\infty \underset{.}{s} \Big(\int_0^N (t+s)^{2(d-1)} \mathrm{e}^{-2\lambda_N(t+s)} \underset{.}{t}\Big)^{p/2} + N \Big(\int_0^N t^{2(d-1)} \mathrm{e}^{-2\lambda_N t} \underset{.}{t}\Big)^{p/2} \Big\} \\
&\equiv CN^{-p/2}\big\{ I_{N1} + I_{N2}\big\}.
\end{aligned}
$$

Here, $I_{N2} \le N\Big(\int_0^\infty t^{2(d-1)}\mathrm{e}^{-2\lambda_N t}\underset{.}{t}\Big)^{p/2} = CN\lambda_N^{(p/2)(1-2d)}$ and $I_{N1} \le CN^{p/2}\int_N^\infty s^{(d-1)p}\mathrm{e}^{-\lambda_N p s}\underset{.}{s} \le CN^{p/2}\lambda_N^{(1-d)p-1}$. Therefore,

$$
\mathbb{E}|Q_{N1}|^{p/2} \le C\big(\lambda_N^{(1-d)p-1} + N^{1-p/2}\lambda_N^{(p/2)(1-2d)}\big). \tag{6.20}
$$

Next,

$$
\begin{aligned}
\mathbb{E}|Q_{N2}|^2 &\le CN^{-2} \sum_{s_2 < s_1 \le N} \mathbb{E}\big(\sum_{t=1\vee s_1}^{N} \omega_{-d,\lambda_N}(t-s_1)\omega_{-d,\lambda_N}(t-s_2)\zeta(s_2)\big)^2 \\
&\le CN^{-2} \sum_{s_2 < s_1 \le N} \sum_{t=1\vee s_1}^{N} \omega^2_{-d,\lambda_N}(t-s_1)\omega^2_{-d,\lambda_N}(t-s_2) \\
&\le CN^{-2} \sum_{t=1}^{N} \big(\sum_{s \le t} \omega^2_{-d,\lambda_N}(t-s)\big)^2 \\
&\le CN^{-2} \int_0^N \underset{.}{t} \Big(\int_{-\infty}^t (t-s)^{2(d-1)} \mathrm{e}^{-2\lambda_N(t-s)} \underset{.}{s}\Big)^2 \\
&= CN^{-1} \Big(\int_0^\infty s^{2(d-1)}\mathrm{e}^{-2\lambda_N s}\underset{.}{s}\Big)^2 \ \le\ CN^{-1}\lambda_N^{2(1-2d)}.
\end{aligned}
\tag{6.21}
$$

Since $(1-d)p - 1 > (p/2)(1-2d)$ and $1 - (p/2) < 0$, (6.20) and (6.21) prove (6.18) for $d > 1/2$. Next, let $d < 1/2$. Then similarly as above we obtain

$$
\begin{aligned}
\mathbb{E}|Q_{N1}|^{p/2} &\le CN^{-p/2} \sum_{s \le N} \Big(\sum_{t=1\vee s}^{\infty} (t-s)_+^{2(d-1)}\Big)^{p/2} \\
&= CN^{-p/2} \Big\{ N\big(\sum_{t=1}^{N} t^{2(d-1)}\big)^{p/2} + \sum_{s \ge N} \big(\sum_{t=1}^{N} (s+t)^{2(d-1)}\big)^{p/2}\Big\} \\
&\le CN^{1-p/2} + CN^{-p/2}\sum_{s \ge N}(Ns^{2(d-1)})^{p/2} \ \le\ CN^{1-p/2}
\end{aligned}
\tag{6.22}
$$

and

$$
\begin{aligned}
\mathbb{E}|Q_{N2}|^2 &\le CN^{-2} \sum_{s_2 < s_1 \le N} \sum_{t=1\vee s_1}^{N} \omega^2_{-d,\lambda_N}(t-s_1)\omega^2_{-d,\lambda_N}(t-s_2) \\
&\le CN^{-2} \sum_{t=1}^{N} \Big(\sum_{s < t} (t-s)^{2(d-1)}\Big)^2 \ \le\ CN^{-1}.
\end{aligned}
\tag{6.23}
$$

(6.22) and (6.23) prove (6.18) for $d < 1/2$.

Finally, let $d = 1/2$. Then since $p > 2$

$$
\begin{aligned}
\mathbb{E}|Q_{N1}|^{p/2} &\le CN^{-p/2} \sum_{s \le N} \Big(\sum_{t=1 \vee s}^{\infty} (t-s)_+^{-1} \Big)^{p/2} \\
&= CN^{-p/2} \Big\{ N \big(\sum_{t=1}^{N} t^{-1} \big)^{p/2} + \sum_{s \ge N} \big(\sum_{t=1}^{N} (s+t)^{-1} \big)^{p/2} \Big\} \\
&\le CN^{1-p/2} (\log N)^{p/2} + CN^{-p/2} \sum_{s \ge N} (Ns^{-1})^{p/2} \\
&\le CN^{1-p/2} (\log N)^{p/2} \; = \; o(1)
\end{aligned}
\tag{6.24}
$$

while

$$
\mathbb{E}|Q_{N2}|^2 \;\le\; CN^{-2} \sum_{t=1}^{N} \Big(\sum_{s<t} (t-s)^{-1} \mathrm{e}^{-\lambda_N (t-s)} \Big)^2 \;\le\; CN^{-1} (\log \lambda_N)^2. \tag{6.25}
$$

(6.24) and (6.25) prove (6.18) for $d = 1/2$. Proposition 5.1 is proved. □

References

[1] Abramowitz, M., Stegun, I.: Handbook of mathematical functions, ninth edition, Dover, New York (1965).

[2] Astrauskas, A. (1983). Limit theorems for sums of linearly generated random variables. *Lithuanian J. Math.* **23** 127–134.

[3] Balan, R., Jakubowski, A. and Louhichi, S. (2016). Functional convergence of linear processes with heavy-tailed innovations. *J. Theoret. Probab.* **29** 491–526.

[4] Billingsley, P. (1968). *Convergence of Probability Measures.* New York: Wiley.

[5] Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods*, 2nd ed.. New York: Springer.

[6] Dickey, D. and Fuller, W. (1979). Distribution of the estimators for autoregressive time series with a unit root. *JASA* **74** 427–431.

[7] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications, vol. 2.* New York: Wiley.

[8] Giraitis, L., Kokoszka, P. and Leipus, R. (2000). Stationary ARCH models: dependence structure and central limit theorem. *Econometric Theory* **16** 3–22.

[9] Giraitis, L., Kokoszka, P., Leipus, R. and Teyssière, G. (2003). Rescaled variance and related tests for long memory in volatility and levels. *J. Econometrics* **112** 265–294.

[10] Giraitis, L., Koul. H.L. and Surgailis, D. (2012). *Large Sample Inference for Long Memory Processes.* London: Imperial College Press.

[11] Gradshteyn, I.S. and Ryzhik, I.M. (2000). *Tables of Integrals and Products.* 6th edition. New York: Academic Press.

[12] Granger, C.W.J. and Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *J. Time Series Anal.* **1** 15–29.

[13] Hosking, J.R.M. (1981). Fractional differencing. *Biometrika* **68** 165–176.

[14] Kasahara, Y. and Maejima, M. (1988). Weighted sums of i.i.d. random variables attracted to integrals of stable processes. *Probab. Theory Relat. Fields* **78** 75–96.

[15] Kokoszka, P.S. and Taqqu, M.S. (1995). Fractional ARIMA with stable innovations. *Stochastic Process. Appl.* **60** 19–47.

[16] Lavancier, F., Leipus, R., Philippe, A. and Surgailis, D. (2013). Detection of non-constant long memory parameter. *Econometric Theory* **29** 1009–1056.

[17] Lo, A. (1991). Long-term memory in stock market prices. *Econometrica* **59** 1279–1313.

[18] Meerschaert, M.M. and Sabzikar, F. (2013). Tempered fractional Brownian motion. *Statist. Probab. Lett.* **83** 2269–2275.

[19] Meerschaert, M.M. and Sabzikar, F. (2016). Tempered fractional stable motion. *J. Theoret. Probab.* **29** 681–706.

[20] Meerschaert, M.M., Sabzikar, F., Phanikumar, M.S. and Zeleke, A. (2014). Tempered fractional time series model for turbulence in geophysical flows. *J. Stat. Mech. Theory Exp.* **2014** P09023.

[21] Phillips, P.C.B. (1987). Time series regression with a unit root. *Econometrica* **55** 277–301.

[22] Sabzikar, F. and Surgailis, D. (2017). Tempered fractional Brownian and stable motions of second kind. Preprint. Available on http://arxiv.org/abs/1702.07258

[23] Samorodnitsky, G. and Taqqu, M.S. (1996). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Boca Raton etc: Chapman and Hall.

[24] Sowell, F. (1990). The fractional unit root distribution. *Econometrica* **58** 495–505.